Scaling of heterogeneous distributions of conductances: Renormalization versus exact results

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In this work we are concerned with the calculation of effective conductance of heterogeneous media. We are interested in determining the conductance when the system becomes macroscopically homogeneous, and the *disorder length* (ξ_D), i.e., the length scale necessary to reach this effective value. Numerical computation of conductances in two-dimensional lattices is done using an exact numerical method and two different renormalization procedures. The conductance values are extracted from (a) power-law, (b) log-normal, or (c) Weibull probability distributions in the interval [0,1]. A parameter μ^{-1} is used to measure the degree of heterogeneity of all three distributions. For the power-law distribution, ξ_D diverges as $\mu\rightarrow 0$ with the exponent ν of the percolation correlation length on the same geometrical support. The log-normal and Weibull distribution reduce to a percolation distribution function, $P(g)=p\,\delta(g-1)+(1-p)\,\delta(g)$, in the limit $\mu\rightarrow 0$. The disorder length remains finite or diverges depending on whether the effective occupation probability p is above the percolation threshold or not. The analysis carried out here may be generalized to a large number of long-tailed distributions, for which percolation ideas apply. [S1063-651X(98)05807-3]

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I. INTRODUCTION

Scaling of transport properties in porous media is important in hydrology, soil science, and, in particular, it has attracted attention in the area of reservoir engineering (see [1], and references therein). Many different scales are involved in this field: oil reservoir characterization is based on a detailed fine-scale description of geological formations that requires handling large data sets. On the other hand, geostatistics predicts property values (i.e., porosity, permeability) based on several sources of data that range from cores (a few centimeters), well logs (from 30 cm to a few meters), and seismic information that spans up to kilometer scales. Reservoir simulation, however, has traditionally worked with smaller sets of data, limited by computing power, and hence a coarser-scale description is required.

Computational grids in reservoir simulations demand assigning local values of the transport properties, e.g., absolute permeability or permeance [2], to the simulation blocks. Such assignments involve propagating ''microscopic'' information to a coarser scale. This is in essence the problem of scaling. The connection between scales is nevertheless nontrivial since natural formations exhibit heterogeneity values that span a large number of characteristic sizes. We will refer to this as strong disorder.

The method employed to scale up transport properties has to average out microscopic quantities in such a way that the effective transport corresponds to the macroscopic detailed solution of the flow problem. The traditional method to determine the upscaled permeability is to take either a simple arithmetic, harmonic, or geometric mean. Among all the mean values, the geometric mean gives the best estimate for isotropic, random distributions [3]. Although this is a method frequently used in the oil industry, it does not provide good estimates for many distribution functions. Analytical methods that rely on perturbation approximations [4] or Kirkpatrick's effective-medium approximation [5] work well only in the case of weak disorder, when fluctuations remain small. Futhermore, mean field approaches assume that the transport property is already homogeneous at the smallest scale, i.e., an intensive quantity can always be defined. Numerical approaches can be employed to obtain an appropriate effective value. However, the expense of computing the effective quantity is usually too high [6].

Renormalization group (RG) has been proposed as an alternative procedure for finding the effective permeability [7]. The technique has been employed successfully for narrow distributions, generally performing better than perturbation approximations [7,8]. Logarithmically broad distributions have been studied as well but with a small variance or an artificial cutoff [7,9]. In addition, RG has been used to scale relative permeability [10]. The method can be adapted to account for strong anisotropy in the distribution of conductances [3]. The technique implies local averaging, but inclusion of more degrees of freedom (larger-cell renormalization) in each step of renormalization improves the accuracy of the method [11].

In spite of the success of the renormalization procedure, it has its pitfalls which merit close scrutiny. The lack of clear understanding of the scope of this technique involves two aspects. (a) Practitioners of renormalization do not often carefully determine the number of steps necessary to obtain the asymptotic value of the hydraulic conductance. (b) It is difficult to know, beforehand, the accuracy of the upscaled conductance. The question of what happens when the conductance distribution is broadened remains open.

Angulo and Medina [2] dealt with the first of our concerns regarding the number of steps required to reach a homogeneous value of conductance for a broad distribution. They carried out a renormalization analysis of the power-law distribution, $P(g)\sim g^{\mu-1}$, on hierarchical lattices, for which renormalization procedures give exact results. Their findings indicate that for lattices of effective dimension $d_e=2$, $\mu=0$ represents a departure from the Gaussian basin, beyond which the system is self-averaging. For $\mu \neq 0$, a disorder length ξ_D can be defined. ξ_D establishes a system size for which the system becomes effectively homogeneous. However, as μ approaches $\mu=0$, ξ_D diverges with the exponent ν of the correlation length of ordinary percolation [12].

In this work, we generalize the findings of Angulo and Medina to regular Euclidean lattices. For the power-law distribution, we compute the effective conductance of two-dimensional (2D) networks by using two RG procedures and an exact numerical method. Obtaining exact results allows us to deal with the issue of the accuracy of renormalization for computing the effective conductance. Additionally, we explore the scaling behavior of two broad distributions common to applications, the log-normal [7] and Weibull [13]. These distributions are also characterized by a parameter, μ^{-1}, that measures the strength of disorder, i.e., the strength of the distribution tail. We seek the existence of the disorder length ξ_D for these distributions. Critical behavior ($\xi_D \to \infty$) is expected in the vicinity of $\mu=0$.

The paper is comprised of five sections. In the second section, we explain the renormalization schemes followed here, namely, King's [7] and the Migdal-Kadanoff [11] procedure. King's renormalization was the starting point for application of this technique in the area of transport in porous media. On the other hand, Migdal-Kadanoff is a simple renormalization procedure used commonly in magnetic systems. In the third section, we describe the characteristics of the distribution functions employed. Then, the results of applying renormalization to the three distribution functions on square networks are reported. Finally, we close the article with a discussion of the results and conclusions, with some emphasis on the applicability of renormalization for computing effective permeabilities.

II. EFFECTIVE CONDUCTANCE

Renormalization group theory has been used to compute the effective conduction properties of random-resistor networks near the percolation threshold p_c since the mid-1970s [14,15]. The main interest resided in the determination of the critical exponents of the percolation transition. The effective-medium approach was already known to fail, below the upper critical dimension, as it cannot account for large fluctuations in the vicinity of p_c [12].

The main idea of RG, in the context of computing effective conduction, is to carry out a partial local average that reduces the number of degrees of freedom accompanied by a transformation of scale. In terms of effective quantities, repeated application of the transformation is expected to reduce the magnitude of the fluctuations, i.e., the distribution is assumed to approach a Gaussian-like function of decreasing variance. The method is computationally efficient with relatively low memory cost. However, it is an approximate solution scheme, except for self-similar networks.

For the purpose of computing the effective conductance, we assume that the block permeability at the lowest scale can be mapped onto a resistor network. This way, the entire problem boils down to computing the effective conductance of the network. We now describe the two procedures selected as the renormalization schemes: King's renormalization [7] (KR) and Migdal-Kadanoff (MK) renormalization [11].

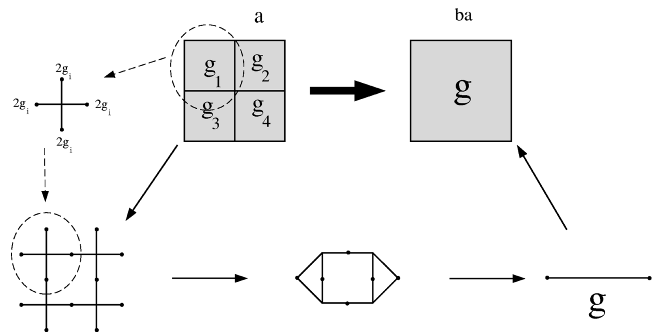

FIG. 1. King's renormalization group procedure. During a preliminary step, block conductances of linear size a are transformed into bond conductances. Renormalization is performed to find a coarser-scale value g. The latter value is assigned to the renormalized block conductance of linear size $b\times a$. The equivalence between block value and bond conductance is indicated in the drawing.

The first step of King's method of renormalization is to replace block conductances, whose values are drawn from a microscopic distribution function, by an equivalent resistor network. Local boundary conditions are applied to a 2×2 block in two dimensions, for which a constant pressure drop is set along the direction of interest. The approximation in King's procedure is introduced by considering that there is no net flow perpendicular to the direction of interest. The procedure is summarized in Fig. 1. The renormalized conductance in two dimensions is computed as follows:

$$g' = 4(g_1+g_3)(g_2+g_4)[g_2g_4(g_1+g_3) + g_1g_3(g_2+g_4)]/g_e,$$

$$g_e \equiv [g_2g_4(g_1+g_3)+g_1g_3(g_2+g_4)][g_1+g_2+g_3+g_4] + 3(g_1+g_2)(g_3+g_4)(g_1+g_3)(g_2+g_4). \quad (1)$$

In the expressions above, g_i ($i=1,\ldots,4$) is the permeance or hydraulic conductance of each lower-scale block and g' is the renormalized conductance. Similar recursion formulas can be easily derived for three-dimensional grids.

The second procedure, MK renormalization, was originally developed from a variational principle applied to magnetic systems [11]. The procedure is carried out recursively in two steps as shown in Fig. 2. The first step consists of moving the inner bond conductances to form combinations in parallel accounting for some of the original conductance of the square lattice. The second step, the decimation step, involves combinations of conductances in series resulting from the parallel arrangement during the bond-moving step. This results in a very efficient algorithm whose basic recursion equation is

$$g' = \left(\frac{1}{g_1+g_2} + \frac{1}{g_3+g_4} \right)^{-1}. \quad (2)$$

The recursion expression can be easily generalized to three dimensions. The simplicity of MK is apparent by comparing Eqs. (1) and (2). This makes the MK procedure very attractive for its computational efficiency.

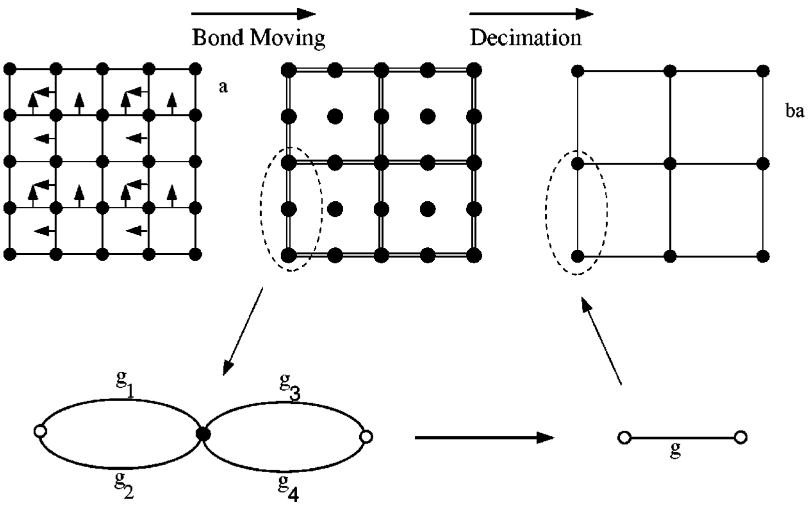

FIG. 2. Migdal-Kadanoff's renormalization group procedure. The bond conductance network is indicated in the drawing. The bond-moving step leads to parallel combinations of conductances (g_1,g_2) and (g_3,g_4). The decimation step combines those in series to yield the effective value g at the coarse-grained scale.

Percolation theory will be essential to understand the results. Each of the renormalization methods leads to a hierarchical structure whose percolation threshold p_c and the corresponding correlation length exponent ν can be calculated from theory. These quantities do not necessarily match those of the square lattice, as displayed in Table I. In the following, the hierarchical structures for the MK and KR renormalization procedures are called MK and KR networks, respectively.

III. DISTRIBUTION FUNCTIONS

A discussion of the main features of the power-law, Weibull, and log-normal distributions is now pertinent, with emphasis in discerning the form of these functions as $\mu \to 0$. The three distributions were mapped onto the interval $[0,1]$. This is equivalent to using distributions of resistances in the $[1,\infty)$ range. It is in this range where the strength of the disorder is more easily observed by following the form of the distribution tails, which determines the moments that may even in fact diverge.

When the distributions are employed in the $[0,1]$ region, all moments of the distributions are finite. However, when seen in log-log scales, the distribution functions exhibit density values along many decades of the conductance axis. This behavior reflects the properties of the corresponding resistance distribution. We will use the power-law distribution to model extreme disorder. The specific function chosen is

$$P_{\mathrm{PL}}(g)=\mu g^{\mu-1}, \quad 0 \leqslant g \leqslant 1. \tag{3}$$

The distribution has three distinct behaviors. It tends to $\delta(g-1)$ for $\mu \gg 1$ and reduces to a uniform random distribution for $\mu=1$. Our interest focuses on the region $0<\mu<1$. In this range, $P_{\mathrm{PL}}(g)$ exhibits a high concentration of low conductance values that grows in a power-law fashion towards $g=0$.

For the log-normal distribution, $\tilde{P}_{\mathrm{LN}}(r)=\sqrt{\mu/\pi}e^{-\ln^2 r}/r$ for $r \in [0,\infty)$, an appropriate change of variable, $r \to 1/g-1$, leads to the following:

$$P_{\mathrm{LN}}(g)=\sqrt{\frac{\mu}{\pi}}\frac{e^{-\mu \ln^2[(1-g)/g]}}{(1-g)g}, \quad 0 \leqslant g \leqslant 1. \tag{4}$$

A similar change of variable for the Weibull distribution, $\tilde{P}_W(r)=\mu e^{-r^\mu} r^{\mu-1}$ for $r \in [0,\infty)$, gives

$$P_W(g)=\mu \frac{e^{-[(1-g)/g]^\mu}(1-g)^{\mu-1}}{g^{\mu+1}}, \quad 0 \leqslant g \leqslant 1. \tag{5}$$

The variance of these distributions was replaced by an equivalent parameter μ, such that $\mu \to 0$ drives the function to the largest contrast in conduction on the networks. This fact is linked to the percolation characteristics of the distribution functions.

The interesting features of the effective conductance of the lattice stem from the binary limiting distribution of the log-normal and Weibull cases. The existence of a limiting binary distribution can be explained based on simple arguments. For all values of μ, the log-normal distribution is symmetric with respect to $g=1/2$. It is observed that in the limit $\mu \to 0$, $P_{\mathrm{LN}}(g)$ vanishes for intermediate values of g, and diverges at $g=0$ and $g=1$. Therefore as $\mu \to 0$, $P_{\mathrm{LN}}(g)$ is essentially nonzero only in the vicinity of either $g=0$ or $g=1$. The width of the resulting peaks reduces monotonically with decreasing values of μ, and hence the height of those peaks grows without bound to keep the distribution normalized. Since the function ought to be positive, the only

TABLE I. Percolation threshold (p_c) and correlation length exponent (ν) of the different lattices. The values were determined theoretically from renormalization.

Lattice	p_c	ν
Square	$\frac{1}{2}$	$\frac{4}{3}$
King's	$\frac{\sqrt{5}-1}{2} \approx 0.618$	$\frac{\log_{10} 2}{\log_{10}(6-2\sqrt{5})} \approx 1.635$
Migdal-Kadanoff	$\frac{3-\sqrt{5}}{2} \approx 0.382$	$\frac{\log_{10} 2}{\log_{10}(6-2\sqrt{5})}$
Berker [a]	$\frac{\sqrt{5}-1}{2}$	$\frac{\log_{10} 2}{\log_{10}(6-2\sqrt{5})}$

[a]Berker renormalization is included because it will be used later for the discussion of results.

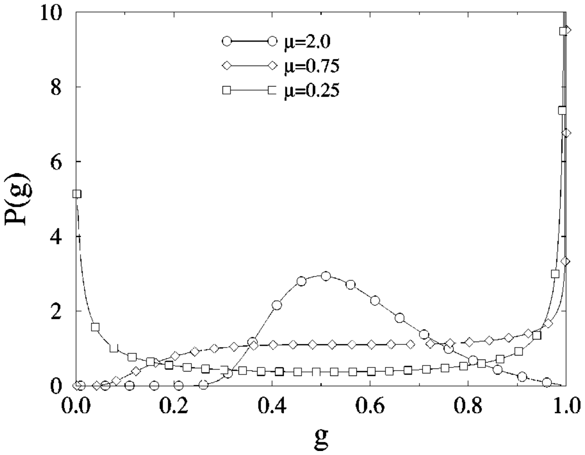

FIG. 3. Weibull distribution as a function of μ. As μ is lowered, the distribution becomes a binary distribution with peaks at $g=0$ and $g=1$.

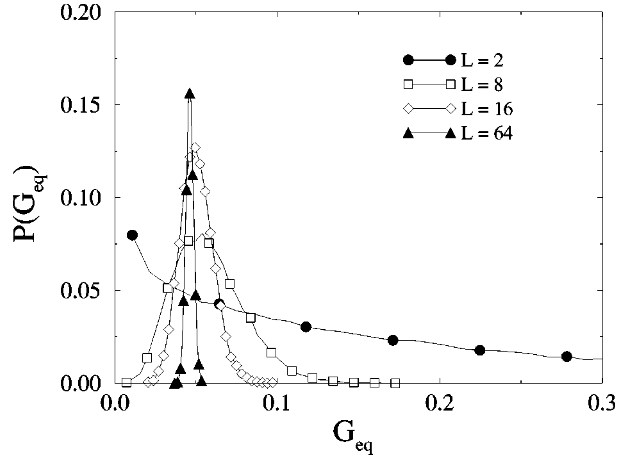

FIG. 4. Approach of the power-law distribution to a Gaussian-like distribution as the system size $L\rightarrow\infty$. G_{eq} was computed for the FL algorithm with $\mu=0.25$.

option is to have a binary distribution, with effective occupation probability $p=0.5$, given that the function is symmetric. By the same sort of arguments a similar observation can be drawn for the Weibull distribution, in the vicinity of $\mu=0$, except that the probability of occupation of nonzero conductances is $p=1-e^{-1}\approx 0.632$ (surmised from numerics). The limiting distribution is then

$$\lim_{\mu\rightarrow 0} P(g)=(1-p)\,\delta(g)+p\,\delta(g-1). \tag{6}$$

This, in fact, is the binary distribution used in ordinary percolation [12]. Figure 3 illustrates this for the Weibull distribution. These particular values of p determine the effective permeability as a function of the renormalization method as $\mu\rightarrow 0$. For instance, if p is below the percolation threshold of a given lattice, renormalization will drive the effective conductance to zero.

IV. RESULTS

The behavior of the effective conductance and disorder length for 2D systems will be shown as functions of the system size and disorder strength. To compare renormalization results, we computed a numerical exact solution based on the algorithm developed originally by Frank and Lobb [16] (FL) using the ∇-Y transformation. Here, we will refer to the results of the FL algorithm as exact. This method has a computational cost proportional to $N\log N$, in contrast, renormalization has a cost proportional to N (N is the number of bonds). All length scales are in units of the lattice constant and conductances are made dimensionless by using the maximum microscopic conductance. The linear system size ranges from $L=2$ up to $L=2^{16}=32\,768$ for renormalization calculations, and considerably smaller lattices ($L_{\max}=2048$) for the FL algorithm. For all the cases, a sufficient number of realizations were carried out to decrease the relative error to less than 1%. The difficulty resulting from handling of very large and very small floating-point numbers forced the development of extended numerical precision routines. Here, G_{eq} is defined as the equivalent conductance of a lattice, whether it is the result of applying the FL algorithm or renormalization procedures. G_{ξ_D} is the value of G_{eq} for a system size $L\geqslant\xi_D$. From the definition of ξ_D, the latter means that the system is effectively homogeneous.

For the log-normal and Weibull distributions, the crucial parameter is the probability of occupation, p. Because the log-normal distribution has a value of $p=0.5$ [Eq. (6)], the KR does not lead to a connected network. This is true as the percolation threshold of the KR lattice is $p_c\approx 0.618>p$. The latter means that the effective conductance will vanish for small values of μ. In contrast, the Migdal-Kadanoff lattice being well above percolation, with $p_c\approx 0.382<p$, yields a finite value of G_{ξ_D}. These two opposite behaviors contrast with the exact result, since the bond network is right at the threshold. Results for the Weibull distribution are easily predicted by knowing that $p=1-e^{-1}$, which is greater than the percolation threshold of any of the networks used here. A rapid convergence to a homogeneous system is expected for this distribution, perhaps with the exception of KR renormalization which is close to the percolating limit of its lattice for this value of p.

Power-law case. Figure 4 shows the upscaled distribution using the FL algorithm, for a particular value of μ. The coarser-scale distribution on the square network shows that averaging small blocks leads to a narrower distribution, even for small values of μ. This confirms that a Gaussian-like distribution is the limiting one for the power law. The decreasing value of the variance of the renormalized distribution is also evidence of the existence of a disorder length ξ_D. This length scale depends on μ (for $\mu\rightarrow 0$) as $\xi\sim\mu^{-\nu}$, where ν is the correlation length exponent of ordinary percolation [2]. Hence, one can collapse the conductance of the network, by rescaling G_{eq} and L by G_{ξ_D} and ξ_D, respectively. This collapse was achieved by using the correlation length exponent of percolation in 2D ($\nu=4/3$), as Fig. 5(a) shows. A diverging value of ξ_D is also found for the MK as well as KR renormalization. This is shown in Figs. 5(b) and 5(c). The overlap is consistent with the correlation length exponent for the MK and KR networks ($\nu=1.635$). From the above, one can generalize the conclusions derived for hierarchical structures [2] to Euclidean lattices.

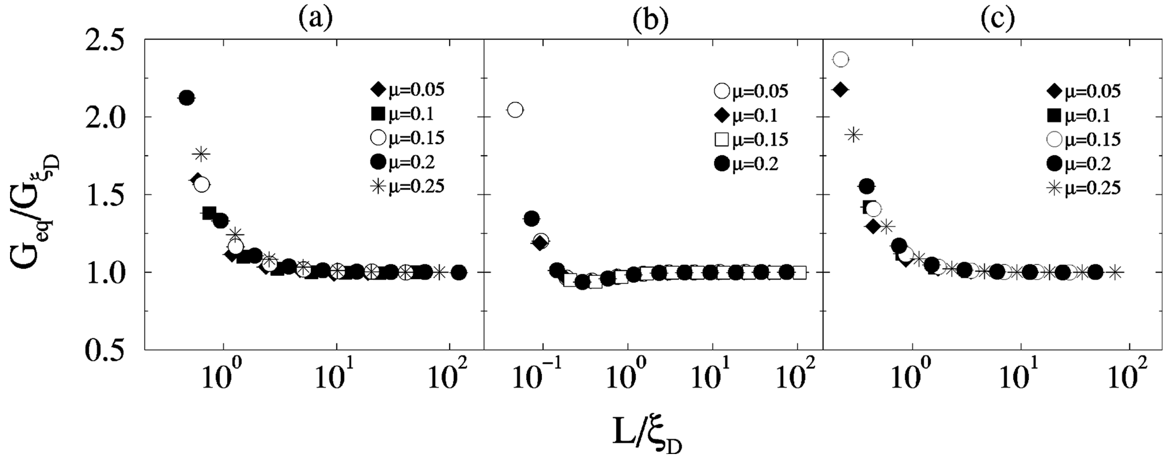

FIG. 5. Normalized conductance $G_{\mathrm{eq}}/G_{\xi_D}$ versus L/ξ_D, where $\xi_D \sim \mu^{-\nu}$. (a) Frank and Lobb's algorithm, (b) Migdal-Kadanoff procedure, and (c) King's procedure. Choosing the corresponding correlation length exponent in each case gives an excellent collapse. $\nu = 4/3$, 1.635, and 1.635 for cases (a), (b), and (c), respectively.

A prediction made by Angulo and Medina [2], based on arguments of Ambegaokar *et al.* [17], involves the dependence of G_{ξ_D} on μ as

$$G_{\xi_D} \propto (1-p_c)^{1/\mu}, \tag{7}$$

where p_c is the percolation threshold of the network. The arguments are valid for broad distributions. A plot of $\log_{10}(G_{\xi_D})$ with $1/\mu$ should yield a straight line whose slope is $\log(1-p_c)$. This prediction can be verified for the FL algorithm and renormalization methods. Figure 6 depicts a very good agreement of the results with the theoretical prediction. In fact, the values of p_c obtained by carrying out a regression analysis are 0.493, 0.617, and 0.381 for the square network, KR procedure, and MK procedure, respectively. All these results deviate less than 2% from the theoretical values. To further verify the validity of Eq. (7), similar computations of G_{eq} were carried out for Berker and Ostlund's network [18], which has the same percolation threshold of King's lattice and the same value of ν. It is striking that the curve obtained using KR lies so close to that determined for Berker's network (see Fig. 6) whose computational cost is similar to that of MK.

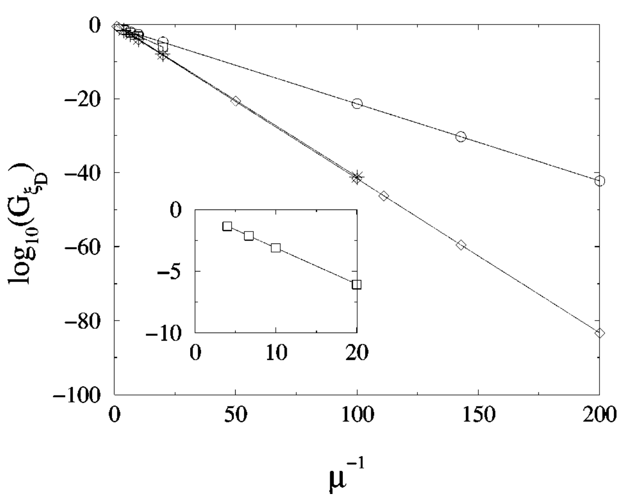

FIG. 6. Logarithm of the conductance beyond the disorder length vs μ^{-1} for the power-law distribution. (□) FL algorithm, (○) Migdal-Kadanoff procedure, (*) King's method, (◇) Berker renormalization. Lines correspond to the fit to Eq. (7). The inset is an enlargement of the left upper region of the plot, which shows the exact result only.

Log-normal case. Let us recall that this distribution tends to a double-δ form with $p=0.5$, as $\mu \rightarrow 0$ [Eq. (6)]. This value of p coincides with the percolation threshold (p_c) of the square lattice, and hence in the limit $\mu \rightarrow 0$, the effective conductance of the log-normal distribution on the square network should exhibit the behavior of the system at p_c. For the purpose of comparison, we directly computed the equivalent conductance of a square lattice for the binary distribution in Eq. (6) with $p=0.5$. Figure 7 indicates that indeed the values of G_{eq} progressively get closer to the percolation line (straight line in the figure) as $\mu \rightarrow 0$. The definition of ξ_D

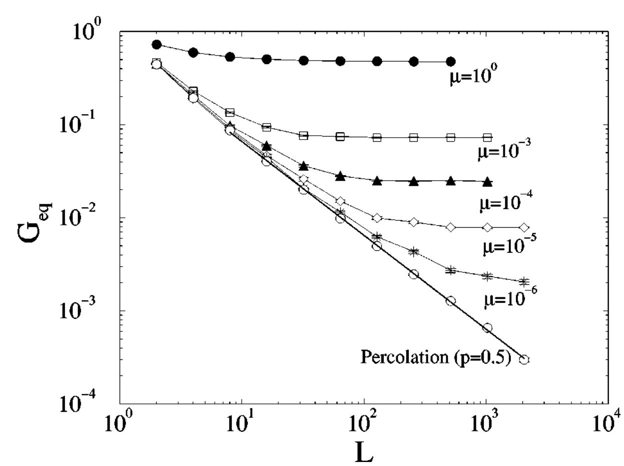

FIG. 7. The behavior of G_{eq} approaches percolation as $\mu \rightarrow 0$ for the log-normal distribution using the FL algorithm. The heavy line is a linear fit of the percolation results (○).

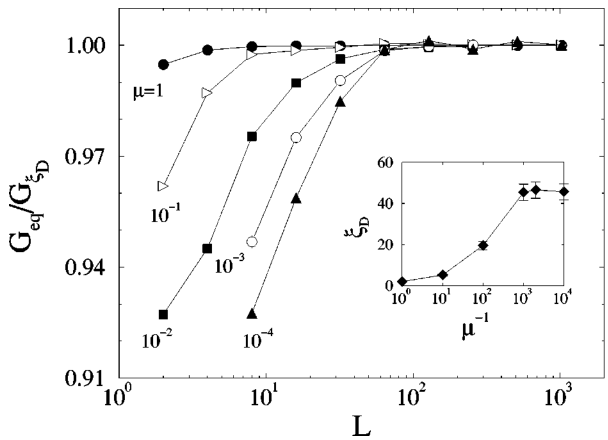

FIG. 8. Normalized G_{eq} vs L for the log-normal distribution using the Migdal-Kadanoff renormalization. The inset illustrates how ξ_D saturates for small values of μ.

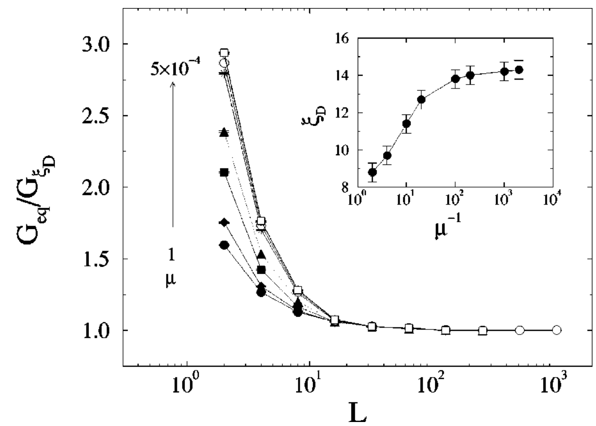

FIG. 10. Normalized G_{eq} vs L for the Weibull distribution using the FL algorithm. The inset illustrates how ξ_D saturates for small values of μ.

implies that G_{eq} should reach a constant value for a system size beyond this length scale. Therefore it can be deduced from Fig. 7 that the disorder length in this case diverges when $\mu\rightarrow 0$.

The MK network should percolate at a probability of occupation $p=0.5$, because its threshold value of $p_c<p$. This means that the network should homogenize rapidly with the application of the renormalization recursion equation. At the same time, the disorder length should reach a saturation value. Figure 8 verifies these predictions. Indeed, the conductance reaches its asymptotic value after a few iterations of the renormalization rule. For King's renormalization procedure, things are entirely different. For $p=0.5$, renormalization should drive the conductance of the network to zero, because the network cannot percolate ($p_c=0.618>0.5$).

Figure 9 clearly summarizes the differences among the three procedures for computing G_{ξ_D} when applied to the log-normal distribution. For a mildly disordered network, that is,

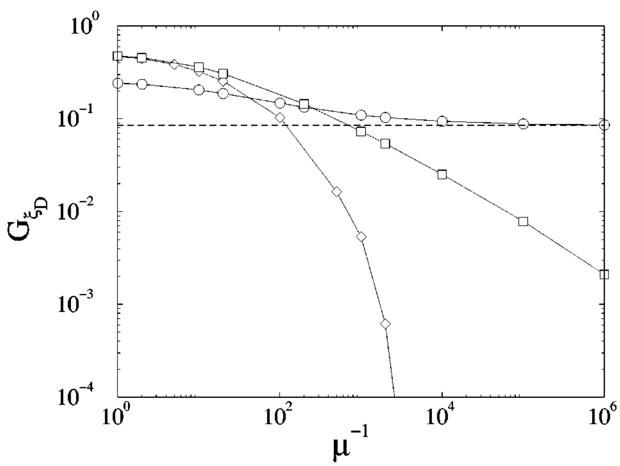

FIG. 9. The effective conductance in the homogeneous limit vs the disorder strength parameter μ for the log-normal distribution function. (○) Migdal-Kadanoff procedure; (□) FL algorithm; (◇) King's renormalization. The dashed line corresponds to the equivalent conductance for percolation with $p=0.5$, under application of the MK recursion equation.

for large values of μ, the three methods do not give exceedingly different values of G_{ξ_D}. As the network becomes more and more disordered, the percolating behavior dominates the form of the curves. Methods that lead to a connected network are driven to a constant finite value of G_{ξ_D}, while those below the percolation threshold are driven to vanishing effective conductance. Interestingly enough, the KR procedure gives reasonable results for values of $\mu>0.1$, as seen from the figure, but as μ is lowered close to zero, the effective conductance vanishes rapidly, while the MK conductance remains finite. The true effective conductance falls in between, but eventually should vanish as $\mu\rightarrow 0$.

Weibull case. This distribution is not expected to exhibit a diverging disorder length provided that the exact and renormalization methods used in this work for calculating the effective conductance lead to a connected network for this distribution. The limiting binary distribution as $\mu\rightarrow 0$ has an occupation fraction $p=1-e^{-1}$. This value is larger than the p_c of the square lattice and for either one of the renormalization networks used in this work. For all cases, the value of G_{ξ_D} is reached after a few applications of the renormalization recursion or equivalently for small sizes of the sample. Figure 10 shows how the effective conductance behaves as μ is decreased. The exact result proves that if the curves are collapsed, the resulting disorder length saturates rapidly (see inset of Fig. 10). Recalling that $p_c=0.5$ explains this rapid convergence. By the same token, the MK network, being well above its percolation threshold, evolves in the same way as the exact result with μ. The KR lattice is close to its value of p_c, and this should have an important effect on the final value of G_{eq}. It can be seen that the effective asymptotic conductance of the KR procedure approaches the value of the true conductance for large values of μ (Fig. 11). As μ becomes smaller, the MK procedure result gets closer to the true conductance, while the KR result grows apart from the true value of G_{ξ_D}.

V. DISCUSSION AND CONCLUSIONS

We have studied the upscaling of hydraulic conductance of three distribution functions in the interval $[0,1]$ on square

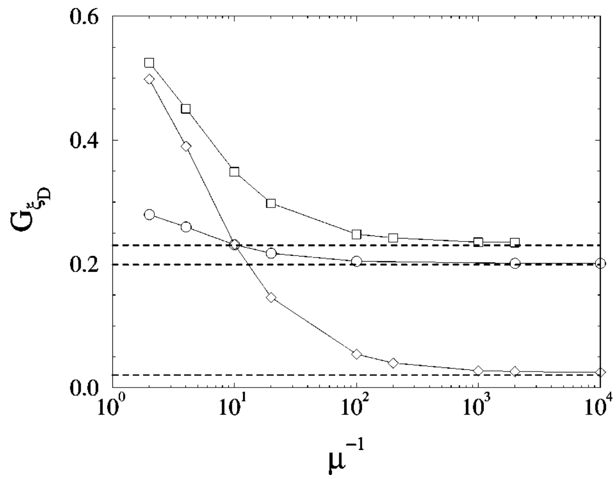

FIG. 11. The effective conductance in the homogeneous limit vs the disorder strength parameter for the Weibull distribution function. (○) Migdal-Kadanoff procedure; (□) FL algorithm; (◇) King's renormalization. The dashed lines correspond to percolation with $p=0.632$ for the three methods.

networks by using an exact method, the FL algorithm, and two approximate renormalization schemes, King's method and Migdal-Kadanoff procedure. The distribution functions, power-law, log-normal, and Weibull distributions exhibit very different features under renormalization.

The main result for the power-law distribution is the existence of a disorder length, on square networks, that behaves in the same fashion as that found by Angulo and Medina [2] on hierarchical lattices. The disorder parameter μ has a critical value $\mu=0$, for networks of effective dimension $d_e=2$. ξ_D diverges as $\mu^{-\nu}$ in the proximity of $\mu=0$, for all methods employed for computing the effective conductance. The divergence exponent of the correlation length of ordinary percolation is consistent with the divergence exponent of the disorder length. This was verified for both renormalization procedures employed in this work, which yield different values of ν. This extends the conclusions of Angulo and Medina to square networks. It is expected that the same holds for many other regular lattices. The dependence of G_{ξ_D} on μ in the critical region agrees very well with the theoretical prediction that $G_{\xi_D}\sim(1-p_c)^{1/\mu}$.

The log-normal distribution goes to a binary distribution with a probability of occupation, $p=0.5$ as $\mu\rightarrow 0$. This makes the analysis of the FL algorithm very interesting, since the limit $\mu=0$ represents the exact percolation threshold of this lattice. This was verified by extensive simulations for large sizes and very small values of μ. The percolation line, represented by the effective conductance of the lattice at percolation, is indeed the limiting value of the log-normal distribution. It turns out that the Migdal-Kadanoff renormalization drives the conductance of the network to a finite value, as the MK lattice is well above the percolation threshold. In this way, the disorder length saturates under this renormalization scheme. King's renormalization, on the other hand, yields a vanishing value of the effective conductance after a few applications of the recursion equation. The two procedures cause departures from the true value of conductance that are on opposite sides of the limiting behavior as $\mu\rightarrow 0$. King's renormalization gives reasonable results for $\mu>0.1$, but fails badly to estimate the correct value of $G_{\rm eq}$ for $\mu<0.1$. This should not come as a surprise given the percolative character of the limiting distribution.

Weibull's limiting distribution has an effective probability of occupation, p, that leads to connected networks for all methods employed here. This means that in the vicinity of $\mu=0$, all procedures give a finite value of G_{ξ_D}, and ξ_D never diverges. However, because the KR hierarchical structure is close to its value of p_c (from below), it tends to have the lowest value of G_{ξ_D}. The KR procedure, that works well in relatively homogeneous systems, does not provide a good estimate for the extreme disorder of the Weibull distribution. With respect to the MK renormalization, although it does not work for all cases, it remains as a better estimate in the limit of extreme disorder.

From the aforementioned discussion, it should be clear that any of the renormalization methods employed here yield reasonable values of $G_{\rm eq}$ for the square network, as long as μ remains sufficiently large. The latter is true for the power-law distribution if μ is larger than 0.1 approximately, whereas for the log-normal and Weibull distributions μ could be somewhat smaller. King's procedure performs, in general, better than Migdal-Kadanoff, with the exception of the power-law distribution (again, only in the limit of non-vanishing μ). However, it is out of the scope of this article to identify a particular method of renormalization, rather we discuss the appropriate choice of the method. As we indicate, the choice refers to the strongest disorder limit, based on percolation analysis.

The results for the Weibull and log-normal cases suggest that the percolation analysis could be extrapolated to other distributions. There is a large number of distribution functions of resistances on the $[0,\infty]$ domain that should exhibit similar characteristics when the conductances are mapped onto the $[0,1]$ interval. For instance, an exponential distribution function, $P(r)=(1/\mu)\exp(-r/\mu)$, goes to $P(g)=(1/\mu g^2)\exp[-(1/\mu)(1/g-1)]$. In the limit $\mu\rightarrow 0$, $P(g)\rightarrow\delta(g-1)$. Long-tail resistance distributions in the $[0,\infty)$ domain should produce a large accumulation of conductance values at $g=0$. Our extreme case is the power law, whose limit is $\delta(g)$. On the other hand, if we were dealing with distributions of conductances in the $[0,\infty)$ region, it should be noticed that the way the random variables are combined resembles that of the resistances. We propose that percolation ideas are directly applicable to this complementary case.

An interesting classical result, found by Bernasconi [15], suggests that the choice of the renormalization method should pay attention to the character of the limiting distribution. Although his renormalization procedure predicted wrong critical exponents, it gave good results for conductivity in a square network at percolation. This result can be explained based on the fact that his renormalization procedure had the same percolation threshold of the original network, and hence its effective conductance vanished for the same value of p. This indicates that if the limiting distribution leads to percolating behavior, the choice of the renormalization method should be based on a similarity of percolation thresholds, or else the effective conductance will be incorrectly predicted.

Our results show that renormalization has to be used very carefully to compute effective conductances. This is particularly true for cases of extreme disorder for which renormalization schemes will drive the system's conductance apart from its true value, either towards zero, or to finite, but wrong, limiting values.

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[1] X-H. Wen and J. J. Gómez-Hernández, J. Hydrol. **183**, ix (1996).
[2] R. Angulo and E. Medina, J. Stat. Phys. **75**, 135 (1994).
[3] Y. Gautier and B. Noetinger, Transp. Porous Media **26**, 1 (1997).
[4] E. K. Paleologos, S. P. Neuman, and D. Tartakovsky, Water Resour. Res. **32**, 1333 (1996).
[5] S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973).
[6] L. J. Durlofsky, Water Resour. Res. **27**, 699 (1991) .
[7] P. R. King, Transp. Porous Media **4**, 37 (1989).
[8] P. R. King, Transp. Porous Media **23**, 337 (1996).
[9] A. Aharony, E. L. Hinrichsen, A. Hansen, J. Feder, T. Jøssang, and H. H. Hardy, Physica A **191**, 410 (1991).
[10] P. R. King, Transp. Porous Media **12**, 237 (1993).
[11] R. J. Creswick, H. A. Farach, and C. P. Poole, Jr., *Introduction to Renormalization Group Methods in Physics* (Wiley Inter-Science, New York, 1992).
[12] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor and Francis, London, 1991).
[13] W. K. Brown and K. H. Wohletz, J. Appl. Phys. **78**, 2758 (1995).
[14] R. B. Stinchcombe and B. P. Watson, J. Phys. C **9**, 3221 (1976).
[15] J. Bernasconi, Phys. Rev. B **18**, 2185 (1978).
[16] D. J. Frank and C. J. Lobb, Phys. Rev. B **37**, 302 (1988).
[17] V. Ambegaokar, B. I. Halperin, and J. S. Langer, Phys. Rev. B **4**, 2612 (1971).
[18] A. N. Berker and S. Ostlund, J. Phys. C **12**, 4961 (1979).